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1.

(与式)

$$=\int_0^{\frac{1}{2}\log 3}\frac{e^x(e^{2x}+4e^x+1)}{(e^{2x}+1)^2}dx$$

$t=e^{2x}+1$

$dt=2e^{2x}dx$

$dx=\frac{1}{2(t-1)}dt$

$$=\int_2^4\frac{\sqrt{t-1}\,(t+4\sqrt{t-1})}{t^2}\cdot\frac{1}{2(t-1)}dt$$

$$=\int_2^4\left\{\frac{\sqrt{t-1}}{2t(t-1)}+\frac{2}{t^2}\right\}dt$$

$$=\frac{1}{2}\int_2^4\frac{1}{t\sqrt{t-1}}dt+\left[-\frac{2}{t}\right]_2^4$$

$u=\sqrt{t-1}$ $\quad dt=2udu$

$u^2=t-1$

$t=u^2+1$

$$=\frac{1}{2}\int_1^{\sqrt{3}}\frac{1}{(u^2+1)u}\cdot 2udu+1-\frac{1}{2}$$

$$=\int_1^{\sqrt{3}}\frac{1}{u^2+1}du+\frac{1}{2}$$

$$=\left[\tan^{-1}u\right]_1^{\sqrt{3}}+\frac{1}{2}$$

$$=\frac{\pi}{3}-\frac{\pi}{4}+\frac{1}{2}$$

$$=\underline{\frac{\pi}{12}+\frac{1}{2}}$$

(2)

$$f'(x)=\frac{x}{(x^2+1)f(x)}$$

$$\Leftrightarrow f(x)f'(x)=\frac{x}{x^2+1}$$

$$\Leftrightarrow 2f(x)f'(x)=\frac{2x}{x^2+1}$$

両辺xで不定積分すると

$$\{f(x)\}^2=\log(x^2+1)+C$$

ところで与式に$x=0$を代入すると

$$f(0)=1$$

より

$$\{f(0)\}^2=C$$

$\therefore C=1$

$$\therefore \underline{f(x)=\sqrt{\log(x^2+1)+1}}$$

2.

(1) $\triangle A_{n_1}A_{n_2}A_3=\frac{\sqrt{3}}{2}$ になるのは

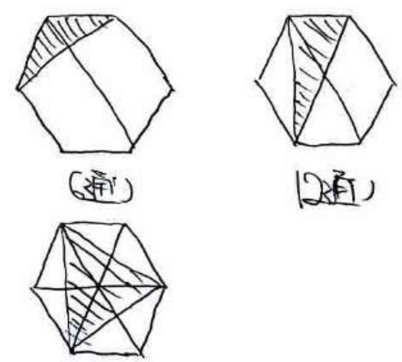
$\triangle A_1A_3A_6$ のような三角形.

これらは全部で $2\times 6=12$コ.

求める条件付き確率は

$$\frac{12\times 3!}{6\cdot 5\cdot 4}=\underline{\frac{3}{5}}$$

(2) 線分が三角形の面積を2等分するのは



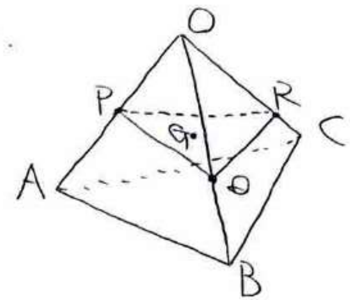
(6通り) (12通り)

$2\times 3=$(6通り)

求める条件付き確率は

$$\frac{6+12+6}{{}_6C_3\times{}_6C_2}=\frac{24}{300}=\underline{\frac{2}{25}}$$

3.



(1)

$$\overrightarrow{OG}=\frac{1}{3}\left(\frac{1}{2}\vec{a}+\frac{2}{3}\vec{b}+\frac{3}{4}\vec{c}\right)$$

$$=\frac{1}{6}\vec{a}+\frac{2}{9}\vec{b}+\frac{1}{4}\vec{c}$$

$\overrightarrow{CH}=k\overrightarrow{CG}$ とおくと

$\overrightarrow{OH}$

$$=\overrightarrow{OC}+\overrightarrow{CH}$$

$$=\vec{c}+k\overrightarrow{CG}$$

$$=\vec{c}+k(\overrightarrow{OG}-\overrightarrow{OC})$$

$$=\frac{k}{6}\vec{a}+\frac{2k}{9}\vec{b}+\underbrace{\left(1-\frac{3}{4}k\right)}_{0}\vec{c}$$

$\therefore k=\frac{4}{3}$

$$\underline{\overrightarrow{OH}=\frac{2}{9}\vec{a}+\frac{8}{27}\vec{b}}$$

(2)

$$\overrightarrow{CH}=\overrightarrow{OH}-\overrightarrow{OC}$$

$$=\frac{2}{9}\vec{a}+\frac{8}{27}\vec{b}-\vec{c}$$

$\overrightarrow{OH}\cdot\vec{a}$
$= \frac{2}{9}|\vec{a}|^2 + \frac{8}{27}\cdot 3 - 1 = 0$
$\therefore |\vec{a}|^2 = \frac{1}{2}$
$|\vec{a}| = \frac{1}{\sqrt{2}}$ //

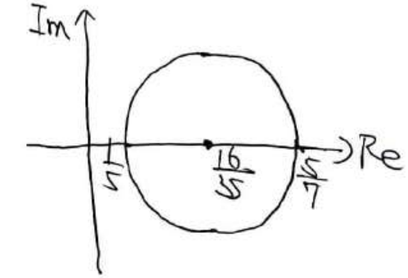
$\overrightarrow{OH}\cdot\vec{b}$
$= \frac{2}{9}\cdot 3 + \frac{8}{27}|\vec{b}|^2 - 9 = 0$
$\therefore |\vec{b}|^2 = \frac{225}{8} \quad |\vec{b}| = \frac{15\sqrt{2}}{4}$ //

(3)
$\triangle OAB = \frac{1}{2}\sqrt{\underbrace{|\vec{a}|^2}_{\frac{1}{2}}\underbrace{|\vec{b}|^2}_{\frac{225}{8}} - \underbrace{(\vec{a}\cdot\vec{b})^2}_{9}}$
$= \frac{9}{8}$ //

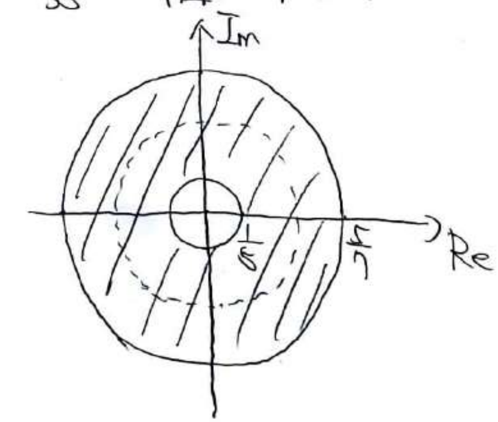
(4)
$|\overrightarrow{OH}|^2 = \left|\frac{2}{9}\vec{a} + \frac{8}{27}\vec{b} - \vec{c}\right|^2$
$= \cdots$
$= \frac{1}{9}$
$\therefore |\overrightarrow{OH}| = \frac{1}{3}$
(四面体OABC) $= \frac{9}{8} \times \frac{1}{3} \times \frac{1}{3}$
$= \frac{1}{8}$ //

4. (1)
$w(1-\bar{\alpha}z) = \alpha - z$
$\iff (1-\bar{\alpha}w)z = \alpha - w$
$z = \frac{\alpha - w}{1-\bar{\alpha}w}$
$\downarrow |z| = \frac{1}{3}$
$\frac{|\alpha - w|}{|1-\bar{\alpha}w|} = \frac{1}{3}$
$3|\alpha - w| = |1-\bar{\alpha}w|$
$\downarrow$ 2乗
$9(\alpha-w)(\bar{\alpha}-\bar{w}) = (1-\bar{\alpha}w)(1-\alpha\bar{w})$
$9(|\alpha|^2 - \alpha\bar{w} - \bar{\alpha}w + |w|^2)$
$= 1 - \alpha\bar{w} - \bar{\alpha}w + |\alpha|^2|w|^2$
$(9-|\alpha|^2)|w|^2 - 8\alpha\bar{w} - 8\bar{\alpha}w = 1 - 9|\alpha|^2$
$|w|^2 - \frac{8\alpha}{9-|\alpha|^2}\bar{w} - \frac{8\bar{\alpha}}{9-|\alpha|^2}w = \frac{1-9|\alpha|^2}{9-|\alpha|^2}$
$\left|w - \frac{8\alpha}{9-|\alpha|^2}\right|^2 = \frac{1-9|\alpha|^2}{9-|\alpha|^2} + \frac{64|\alpha|^2}{\{9-|\alpha|^2\}^2}$
$= \frac{\{3(1-|\alpha|^2)\}^2}{\{9-|\alpha|^2\}^2}$
$\therefore \left|w - \frac{8\alpha}{9-|\alpha|^2}\right| = \frac{3(1-|\alpha|^2)}{9-|\alpha|^2}$

Dは中心 $\frac{8\alpha}{9-|\alpha|^2}$, 半径 $\frac{3(1-|\alpha|^2)}{9-|\alpha|^2}$
の円. $\alpha = \frac{1}{2}$のとき
$\left|w - \frac{16}{35}\right| = \frac{9}{35}$

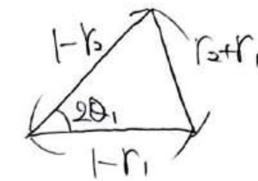


(2) $|\alpha| = \frac{1}{2}$のとき
$\left|w - \frac{32}{35}\alpha\right| = \frac{9}{35}$
wは中心 $\frac{32}{35}\alpha$, 半径 $\frac{9}{35}$の円
だが, $\left|\frac{32}{35}\alpha\right| = \frac{16}{35}$の円より
$\frac{32}{35}\alpha$は半径が　中心が原点



Eは上の領域(境界含む).

5.
(1)



$(r_2+r_1)^2 = (1-r_2)^2 + (1-r_1)^2$
$-2(1-r_2)(1-r_1)\cos 2\theta_1$
$\iff 4r_1r_2 - 2 = -2(1-r_2)(1-r_1)\cos 2\theta_1$
$\iff \underbrace{\cos 2\theta_1}_{1-2\sin^2\theta_1} = \frac{1-2r_1r_2}{(1-r_1)(1-r_2)}$
$\sin^2\theta_1 = \frac{1}{2}\left\{1 - \frac{1-2r_1r_2}{(1-r_1)(1-r_2)}\right\}$
$= \cdots$
$= \frac{r_1r_2}{(1-r_1)(1-r_2)}$ //

(2)
$\sin^2\theta_n = \frac{r_nr_{n+1}}{(1-r_n)(1-r_{n+1})}$
$\downarrow r_n = \frac{1}{2^{n+1}+1}$
$= \frac{1}{2^{2n+3}}$
$\therefore \sin\theta_n = \frac{1}{2^{n+\frac{3}{2}}}$ //

(3) 省略
(4) 0.35