2024 順天堂大(医)

$\boxed{\text{I}}$

(1)(a)

$A+B=\int_0^{\frac{\pi}{4}} 1\,dx$

$=\frac{1}{4}\pi$

$A-B$

$=\int_0^{\frac{\pi}{4}} \frac{\cos x-\sin x}{\sin x+\cos x}\,dx$

$=\left[\log|\sin x+\cos x|\right]_0^{\frac{\pi}{4}}$

$=\log\sqrt{2}=\frac{1}{2}\log 2$

$A=\frac{1}{8}\pi+\frac{1}{4}\log 2$

$B=\frac{1}{8}\pi-\frac{1}{4}\log 2$

(b)

$3C+2D=\int_0^{\frac{\pi}{2}} 1\,dx=\frac{\pi}{2}$

$2C-3D=\left[\log|2\sin x+3\cos x+13|\right]_0^{\frac{\pi}{2}}$

$=\log\frac{15}{16}$

$C=\frac{3}{26}\pi+\frac{2}{13}\log\frac{15}{16}$

$D=\frac{1}{13}\pi-\frac{3}{13}\log\frac{15}{16}$

(2)

(a)

$1\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}=\frac{625}{1296}$

(b)

$1-\frac{625}{1296}=\frac{671}{1296}$

(c)

P(4回以上連続)

$=P(4連続)+P(5連続)$

$=P\begin{pmatrix}\circ\circ\circ\circ\times\\ \times\circ\circ\circ\circ\end{pmatrix}+\left(\frac{1}{6}\right)^4$

$=\frac{{}_6P_2\times 2}{6^5}+\frac{1}{1296}=\frac{11}{1296}$

(d)

P(3連続)

$=P\begin{pmatrix}\circ\circ\circ\times\triangle\\ \times\circ\circ\circ\triangle\\ \times\triangle\circ\circ\circ\end{pmatrix}$

$=\frac{6\cdot5\cdot6+6\cdot5\cdot5+6\cdot5\cdot6}{6^5}=\frac{85}{1296}$

P(3回以上連続)

$=\frac{85}{1296}+\frac{11}{1296}=\frac{2}{27}$

(3)

$a_n=\sum_{k=0}^{n} b^{n-k}c^k$

(a)

$a_n=\sum_{k=0}^{n} 3^{n-k}(-2)^k$

$a_5=3^5+3^4(-2)+3^3(-2)^2$

$+3^2(-2)^3+3^1(-2)^4+(-2)^5$

$a_5\equiv 3(-1)^2+(-1)^2(-2)+3(-1)4$

$+(-1)(-8)+3-2$

$\equiv 3-2-12+8+1$

$\equiv -2\equiv 3 \pmod 5$

(b)

$a_n=\sum_{k=0}^{n}\left(\frac{1}{2}\right)^{n-k}\left(-\frac{1}{3}\right)^k$

$=\left(\frac{1}{2}\right)^n\sum_{k=0}^{n}\left(-\frac{2}{3}\right)^k$

$=\left(\frac{1}{2}\right)^n\cdot\frac{1-\left(-\frac{2}{3}\right)^{n+1}}{1+\frac{2}{3}}$

$=\frac{3}{5}\left(\frac{1}{2}\right)^n-\frac{3}{5}\left(-\frac{1}{3}\right)^n\left(-\frac{2}{3}\right)$

$=\frac{3}{5}\left(\frac{1}{2}\right)^n+\frac{2}{5}\left(-\frac{1}{3}\right)^n$

$\sum_{n=1}^{\infty}a_n=\frac{3}{5}\cdot\frac{\frac{1}{2}}{1-\frac{1}{2}}+\frac{2}{5}\cdot\frac{-\frac{1}{3}}{1+\frac{1}{3}}$

$=\frac{3}{5}-\frac{1}{10}$

$=\frac{1}{2}$

(c)

$\sum_{n=1}^{\infty}na_n$

$=\sum_{n=1}^{\infty}\left\{\frac{3n}{5}\left(\frac{1}{2}\right)^n+\frac{2n}{5}\left(-\frac{1}{3}\right)^n\right\}$

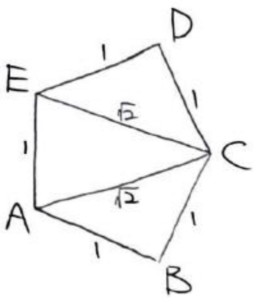
$=\cdots$

$=\frac{9}{8}$

$S_n=\sum_{k=1}^{n}kr^k$

を出して2つに適用するといいです.

II



(a)

余 $\cos\angle ACE = \dfrac{2+2-1}{2\sqrt{2}\cdot\sqrt{2}}$

$= \dfrac{3}{4}$

(正五角形ABCDE)

$= \dfrac{1}{2}\sqrt{2}\sqrt{2}\dfrac{\sqrt{7}}{4} + 1$

$= \dfrac{4+\sqrt{7}}{4}$

$\vec{p}\cdot\vec{q}$

$= 1\cdot 1\cdot\cos\angle BCD$

$= \cos(\angle ACE + 90^\circ)$

$= -\sin\angle ACE = -\dfrac{\sqrt{7}}{4}$

$|\overrightarrow{CA}|^2$

$= s^2 + t^2 - \dfrac{\sqrt{7}}{2}st = 2 \quad \cdots ①$

$\overrightarrow{CA} = \vec{p} + \overrightarrow{BA}$

$\Leftrightarrow \overrightarrow{BA} = (s-1)\vec{p} + t\vec{q}$

$\overrightarrow{BA}\cdot\vec{p} = s - 1 - \dfrac{\sqrt{7}}{4}t = 0 \quad \cdots ②$

①, ②より

$(s, t) = \left(\dfrac{3\pm\sqrt{7}}{3}, \pm\dfrac{4}{3}\right)$

(複号同順)

また

$\overrightarrow{CD}\cdot\overrightarrow{BA}$

$= -\dfrac{\sqrt{7}}{4}(s-1) + t > 0$

$\Leftrightarrow t > \dfrac{\sqrt{7}}{4}(s-1)$

これをみたすのは $s = \dfrac{3+\sqrt{7}}{3}$, $t = \dfrac{4}{3}$

$\overrightarrow{CB} = t\vec{p} + s\vec{q}$

$\overrightarrow{CM} = \dfrac{1}{2}(\overrightarrow{CA} + \overrightarrow{CE})$

$= \dfrac{7+\sqrt{7}}{6}(\vec{p} + \vec{q})$

$\overrightarrow{MB} = \overrightarrow{CB} - \overrightarrow{CM}$

$= -\dfrac{1+\sqrt{7}}{6}\vec{p} - \dfrac{7+\sqrt{7}}{6}\vec{q}$

(c) BはMM'の中点

$\overrightarrow{MM'} = 2\overrightarrow{MB} = -\dfrac{1+\sqrt{7}}{3}\vec{p} - \dfrac{7+\sqrt{7}}{3}\vec{q}$

$|\overrightarrow{MM'}|^2 = \left(\dfrac{1+\sqrt{7}}{3}\right)^2 |\vec{p} + \sqrt{7}\vec{q}|^2$

$= \cdots$

$= 4+\sqrt{7}$

$\overrightarrow{MM'}\cdot\overrightarrow{MM''} = 0$

$\triangle MM'M'' = \cdots = \dfrac{4+\sqrt{7}}{2}$

III

(1) $\underset{y}{f(x)} = \sqrt{x+1}$

$x = \sqrt{y+1}$

$y = x^2 - 1 \quad (x \geq 0)$

$a_n = f^{-1}(n)$

$= n^2 - 1 \quad a_2 = 3, \ a_3 = 8$

(2)

$n \geq 2$で

S_n

$= \displaystyle\sum_{k=1}^{n-1} k\{(k+1)^2 - k^2\}$

$= \displaystyle\sum_{k=1}^{n-1}(2k^2 + k)$

$= \cdots = \dfrac{1}{6}n(n-1)(4n+1)$

$n=1$のときも成立.

$S_n = \dfrac{1}{6}n(n-1)(4n+1)$

(3) $f^{-1}(x) = \log x$

$a_n = \log n$

$n \geq 2$で

S_n

$= \displaystyle\sum_{k=1}^{n-1} k\{\log(k+1) - \log k\}$

$= \displaystyle\sum_{k=1}^{n-1}\{(k+1)\log(k+1) - k\log k - \log(k+1)\}$

$= \displaystyle\sum_{k=1}^{n} k\log k - \sum_{k=1}^{n-1} k\log k - \log(n!)$

$= n\log n - \log(n!)$

これは$n=1$でも成立.

(4)

平均値の定理より

$\dfrac{\log(k+1) - \log k}{k+1-k} = \dfrac{1}{c}$, $k < c < k+1$

をみたすcが存在.より

$\log(k+1) - \log k = \dfrac{1}{c} < \dfrac{1}{k}$

$\therefore S_n < \displaystyle\sum_{k=1}^{n-1} k\cdot\dfrac{1}{k} = n-1$ $(n \geq 2)$

$n=1$で$S_n = 0$より

$S_n \leq n-1 \quad (n \geq 1)$

$\log n^n + \log e^{1-n} \leq \log(n!)$

$\therefore n! \geq n^n e^{1-n}$