2020 慶應 (医)

[I]

(1)

$\alpha : z = px + qy$ とおく

$\downarrow$ A, B 通る

$a = p,\ b = q$

$\therefore \alpha : z = ax + by$

$\downarrow$ xy平面 $(z=0)$

$l : 0 = ax + by$

$\Leftrightarrow y = -\frac{a}{b}x$

これは $\vec{u} = (1, -\frac{a}{b}, 0)$ に平行.

αの法線ベクトル $\vec{n_1} = \begin{pmatrix} a \\ b \\ -1 \end{pmatrix}$.

xy平面の法線ベクトル $\vec{n_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

のなす角を φ とすると

$\vec{n_1} \cdot \vec{n_2} = |\vec{n_1}||\vec{n_2}|\cos\varphi$

$\Leftrightarrow -1 = \sqrt{a^2+b^2+1}\cos\varphi$

$\therefore \cos\varphi = \frac{-1}{\sqrt{a^2+b^2+1}}$

これより $\cos\theta = \frac{1}{\sqrt{a^2+b^2+1}}$

(2)

$f(x) = \frac{\log x}{\sqrt{x}}$

$f'(x) = \frac{\frac{1}{x}\sqrt{x} - \log x \cdot \frac{1}{2}x^{-\frac{1}{2}}}{x}$

$= \frac{2 - \log x}{2x\sqrt{x}}$

x	$0 \cdots e^2 \cdots$
$f'(x)$	$+ \quad 0 \quad -$
$f(x)$	$\nearrow \qquad \searrow$

$x = e^2$ で最大値をとる.

(Dの面積)

$= \int_1^{e^2} \frac{\log x}{\sqrt{x}} dx$

$= \left[2x^{\frac{1}{2}}\log x - 4x^{\frac{1}{2}}\right]_1^{e^2}$

$\boxed{\frac{\log x}{\sqrt{x}} + \frac{2}{\sqrt{x}}}$ $\boxed{-\frac{2}{\sqrt{x}}}$

$= 4e - 4e - (-4) = 4$

(回転体の体積)

$= \int_1^{e^2} \frac{1}{x}(\log x)^2 \pi dx$

$= \left[\frac{\pi}{3}(\log x)^3\right]_1^{e^2}$

$= \frac{8}{3}\pi$

(3) $\alpha = a + bi$ とおく.

$\alpha z = (a+bi)(x+yi)$

$= ax - by + (ay + bx)i$

$\alpha z + \bar{\alpha}\bar{z} + 1$

$= \alpha z + \overline{\alpha z} + 1$

$= 2ax - 2by + 1 = 3x - 4y + 1$

$\therefore a = \frac{3}{2},\ b = 2$

$\therefore \alpha = \frac{3}{2} + 2i$

$\alpha z + \bar{\alpha}\bar{z} + 1 = 0$

$\downarrow z = \frac{1}{w}\ (w \neq 0)$

$\frac{\alpha}{w} + \frac{\bar{\alpha}}{\bar{w}} + 1 = 0$

$\Leftrightarrow \alpha\bar{w} + \bar{\alpha}w + w\bar{w} = 0$

$\Leftrightarrow (w+\alpha)(\bar{w}+\bar{\alpha}) = \alpha\bar{\alpha}$

$\Leftrightarrow |w+\alpha|^2 = |\alpha|^2$

$\therefore |w - (-\alpha)| = |\alpha|$

wは $-\alpha = -\frac{3}{2} - 2i$ を中心,

半径 $|\alpha| = \frac{5}{2}$ の円周上

$|w + \alpha| = |\alpha|$

$\Leftrightarrow |w+\alpha||z| = |\alpha||z|$

$\Leftrightarrow |\alpha z + 1| = |\alpha||z|$

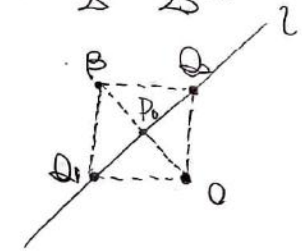
$\Leftrightarrow |z + \frac{1}{\alpha}| = |z|$

$\beta = -\frac{1}{\alpha}$

$= -\frac{2}{3+4i} \times \frac{3-4i}{3-4i}$

$= -\frac{6-8i}{25}$

$= -\frac{6}{25} + \frac{8}{25}i$



図より z_1, z_2 は

$\beta \times \frac{1}{\sqrt{2}}\left\{\cos\left(\pm\frac{\pi}{4}\right) + i\sin\left(\pm\frac{\pi}{4}\right)\right\}$

$= \left(-\frac{6}{25} + \frac{8}{25}i\right)\left(\frac{1}{2} \pm \frac{1}{2}i\right)$

$= \left(-\frac{3}{25} + \frac{4}{25}i\right)(1 \pm i)$

$$z_1=\left(-\frac{3}{25}+\frac{4}{25}i\right)(1+i)$$
$$=-\frac{7}{25}+\frac{1}{25}i$$
$$z_2=\left(-\frac{3}{25}+\frac{4}{25}i\right)(1-i)$$
$$=\frac{1}{25}+\frac{7}{25}i$$

[II]

(1)

P(A,B少なくとも1人が1を引く)

$$=1-P(\text{1が出ない})$$
$$=1-\left(\frac{{}_{n-1}C_2}{{}_nC_2}\right)^2$$
$$=1-\left(\frac{n-2}{n}\right)^2$$
$$=\frac{4n-4}{n^2}$$

(2)

P(A,Bが共通に引くカードが1枚)

$$=1\cdot\frac{n-2}{{}_nC_2}\times 2=\frac{4(n-2)}{n(n-1)}$$

Aは何でも　Aが選んだもの以外の1枚　Aのどちらを共通にするか

(3)

P(Aの2枚が2以下を1枚引く)
+P(　〃　2枚引く)
+P(Bの2枚が2以下を1枚引く)
+P(　〃　2枚引く)

$$=\left\{\frac{2\cdot(n-2)}{{}_nC_2}\cdot\frac{{}_{n-2}C_2}{{}_nC_2}+\frac{1}{{}_nC_2}\cdot\frac{{}_{n-2}C_2}{{}_nC_2}\right\}\times 2$$
$$=\left(\frac{4(n-2)}{n(n-1)}+\frac{2}{n(n-1)}\right)\frac{2(n-2)(n-3)}{n(n-1)}$$
$$=\frac{2(2n-3)}{n(n-1)}\times\frac{2(n-2)(n-3)}{n(n-1)}$$
$$=\frac{4(2n-3)(n-2)(n-3)}{n^2(n-1)^2}$$

(4)

お互いの大きい方をkとする.
求める確率は

$$\sum_{k=2}^{n}\left(\frac{k-1}{{}_nC_2}\right)^2$$
$$=\left(\frac{2}{n(n-1)}\right)^2\sum_{k=2}^{n}(k-1)^2$$
$$=\frac{4}{n^2(n-1)^2}\sum_{k=1}^{n-1}k^2$$
$$=\frac{4}{n^2(n-1)^2}\cdot\frac{1}{6}(n-1)n(2n-1)$$
$$=\frac{2(2n-1)}{3n(n-1)}$$

(5)

A君の小さい方をkとする.
求める確率は

$$\sum_{k=2}^{n-1}\frac{n-k}{{}_nC_2}\cdot\frac{{}_kC_2}{{}_nC_2}$$
$$=\frac{4}{n^2(n-1)^2}\sum_{k=2}^{n-1}(n-k)\frac{k(k-1)}{2}$$
$$=\frac{2}{n^2(n-1)^2}\left\{n\sum_{k=2}^{n-1}k(k-1)-\sum_{k=2}^{n-1}k^2(k-1)\right\}$$
$$=\frac{2}{n^2(n-1)^2}\left\{n\sum_{k=1}^{n-2}k(k+1)-\sum_{k=1}^{n-2}k(k+1)^2\right\}$$
$$=\frac{2}{n^2(n-1)^2}\left\{n\cdot\frac{1}{3}(n-2)(n-1)n-\frac{1}{4}(n-2)^2(n-1)^2-\frac{2}{6}(n-2)(n-1)(2n-3)-\frac{1}{2}(n-2)(n-1)\right\}$$
$$=\frac{2}{n^2(n-1)}\left\{\frac{n^2}{3}(n-2)-\frac{1}{4}(n-2)^2(n-1)-\frac{1}{3}(n-2)(2n-3)-\frac{1}{2}(n-2)\right\}$$
$$=\frac{2(n-2)}{n^2(n-1)}\left\{\frac{n^2}{3}-\frac{1}{4}(n-2)(n-1)-\frac{1}{3}(2n-3)-\frac{1}{2}\right\}$$
$$=\frac{2(n-2)\{4n^2-3(n-2)(n-1)-(8n-12)-6\}}{12n^2(n-1)}$$
$$=\frac{(n-2)(n^2+n)}{6n^2(n-1)}$$
$$=\frac{(n-2)(n+1)}{6n(n-1)}$$

余談

Aのカードをa_1, a_2 $(a_1<a_2)$
Bのカードをb_1, b_2 $(b_1<b_2)$
としたとき

$$1\leqq b_1<b_2\leqq a_1<a_2\leqq n$$
$$\downarrow$$
$$1\leqq b_1<b_2<a_1+1<a_2+1\leqq n+1$$

より選び方は${}_{n+1}C_4$通りと考えた方が圧倒的に速かったです.

つまり $\frac{{}_{n+1}C_4}{({}_nC_2)^2}=\cdots=\frac{(n-2)(n+1)}{6n(n-1)}$

[III]

(1) $f(x)+f\left(\frac{\pi}{2}-x\right)$

$=\frac{1}{\sin^2 x}+\frac{1}{\cos^2 x}$

$=\frac{1}{\sin^2 x\cos^2 x}$

$=\frac{4}{\sin^2 2x}=4f(2x)$

$\underline{a=4}$ //

(2)

S_1

$=f\left(\frac{\pi}{4}\right)=\underline{2}$ //

S_2

$=\sum_{k=1}^{3} f\left(\frac{k\pi}{8}\right)$

$=f\left(\frac{\pi}{8}\right)+f\left(\frac{\pi}{4}\right)+f\left(\frac{3}{8}\pi\right)$

$=\underbrace{4f\left(\frac{2}{8}\pi\right)}_{\because(1)}+f\left(\frac{\pi}{4}\right)$

$=5f\left(\frac{\pi}{4}\right)=\underline{10}$ //

S_3

$=\sum_{k=1}^{7} f\left(\frac{k\pi}{16}\right)$

$=f\left(\frac{\pi}{16}\right)+f\left(\frac{\pi}{8}\right)+f\left(\frac{3}{16}\pi\right)$

$+f\left(\frac{\pi}{4}\right)+f\left(\frac{5}{16}\pi\right)+f\left(\frac{3}{8}\pi\right)$

$+f\left(\frac{7}{16}\pi\right)$

$=4f\left(\frac{2}{16}\pi\right)+4f\left(\frac{6}{16}\pi\right)+10$

$=4\underbrace{\left(f\left(\frac{\pi}{8}\right)+f\left(\frac{3}{8}\pi\right)\right)}_{8}+10$

$=\underline{42}$ //

S_{n+1}

$=\sum_{k=1}^{2^{n+1}-1} f\left(\frac{k\pi}{2^{n+2}}\right)$

$=f\left(\frac{\pi}{2^{n+2}}\right)+f\left(\frac{2\pi}{2^{n+2}}\right)+f\left(\frac{3\pi}{2^{n+2}}\right)$

$+\cdots+f\left(\frac{(2^{n+1}-2)\pi}{2^{n+2}}\right)+f\left(\frac{(2^{n+1}-1)\pi}{2^{n+2}}\right)$

$=4f\left(\frac{\pi}{2^{n+1}}\right)+4f\left(\frac{2\pi}{2^{n+1}}\right)$

$+\cdots+4f\left(\frac{(2^n-1)\pi}{2^{n+1}}\right)+f\left(\frac{\pi}{4}\right)$

$=\underline{4S_n+2}$ //

$S_{n+1}+\frac{2}{3}=4\left(S_n+\frac{2}{3}\right)$

$\Downarrow$

$S_n+\frac{2}{3}=\left(S_1+\frac{2}{3}\right)4^{n-1}$

$\therefore S_n=\frac{8}{3}\cdot 4^{n-1}-\frac{2}{3}$

$=\underline{\frac{2\cdot 4^n-2}{3}}$ //

$g(x)$

$=\frac{1}{\tan^2 x}$

$=\frac{\cos^2 x}{\sin^2 x}$

$=\frac{1-\sin^2 x}{\sin^2 x}$

$=f(x)-1$

T_n

$=\sum_{k=1}^{2^n-1}\left\{f\left(\frac{k\pi}{2^{n+1}}\right)-1\right\}$

$=S_n-(2^n-1)$

$=\underline{\frac{2}{3}\cdot 4^n-2^n+\frac{1}{3}}$ //

$\sin^2\theta<\theta^2<\tan^2\theta$

$\iff \frac{1}{\tan^2\theta}<\frac{1}{\theta^2}<\frac{1}{\sin^2\theta}\ \left(0<\theta<\frac{\pi}{2}\right)$

$\theta=\frac{k\pi}{2^{n+1}}$ とおくと

$g\left(\frac{k}{2^{n+1}}\pi\right)<\left(\frac{2^{n+1}}{k\pi}\right)^2<f\left(\frac{k}{2^{n+1}}\pi\right)$

$\iff g\left(\frac{k\pi}{2^{n+1}}\right)<\frac{4^{n+1}}{\pi^2}\cdot\frac{1}{k^2}<f\left(\frac{k\pi}{2^{n+1}}\right)$

$\iff \frac{\pi^2}{4^{n+1}}g\left(\frac{k\pi}{2^{n+1}}\right)<\frac{1}{k^2}<\frac{\pi^2}{4^{n+1}}f\left(\frac{k\pi}{2^{n+1}}\right)$

$\Downarrow$ $k=1\sim 2^n-1$ まで和

$\frac{\pi^2}{4^{n+1}}T_n<\sum_{k=1}^{2^n-1}\frac{1}{k^2}<\frac{\pi^2}{4^{n+1}}S_n$

$\lim_{n\to\infty}\frac{\pi^2}{4^{n+1}}T_n=\lim_{n\to\infty}\frac{\pi^2}{4^{n+1}}S_n=\frac{\pi^2}{6}$

より

$\lim_{n\to\infty}\sum_{k=1}^{2^n-1}\frac{1}{k^2}=\underline{\frac{\pi^2}{6}}$ //

[IV]

(1)

A
$= \int_0^{\sqrt{c}} b\left(1-\frac{x^2}{c}\right)dx$
$= \left[bx - \frac{b}{3c}x^3\right]_0^{\sqrt{c}}$
$= b\sqrt{c} - \frac{b}{3}\sqrt{c}$
$= \frac{2}{3}b\sqrt{c}$

$\therefore b = \frac{3A}{2\sqrt{c}}$

(2)

$r(x)$
$= \sqrt{x^2 + b^2\left(1-\frac{x^2}{c}\right)^2}$
$= \sqrt{x^2 + \frac{9A^2}{4c}\left(1 - \frac{2x^2}{c} + \frac{x^4}{c^2}\right)}$

$\{r(x)\}^2$
$= \frac{9A^2}{4c^3}x^4 + \left(1 - \frac{9A^2}{2c^2}\right)x^2 + \frac{9A^2}{4c}$

$1 - \frac{9A^2}{2c^2} \geqq 0$
$\Leftrightarrow 2c^2 \geqq 9A^2$
$\therefore c \geqq \frac{3A}{\sqrt{2}}$

このとき$\{r(x)\}^2$, つまり$r(x)$も$0 \leqq x \leqq \sqrt{c}$で増加.

$0 < c < \frac{3A}{\sqrt{2}}$のとき

$x^2 = t$とおくと

$\{r(x)\}^2$
$= \frac{9A^2}{4c^3}t^2 + \left(1 - \frac{9A^2}{2c^2}\right)t + \frac{9A^2}{4c}$

$\frac{d\{r(x)\}^2}{dt}$
$= \frac{9A^2}{2c^3}t + 1 - \frac{9A^2}{2c^2}$

$\frac{d\{r(x)\}^2}{dt} = 0 \Leftrightarrow 9A^2 t = 9A^2c - 2c^3$
$\Leftrightarrow t = c - \frac{2c^3}{9A^2}$

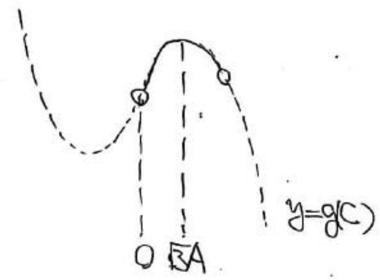
このとき$\{r(x)\}^2$つまり$r(x)$も最小.

$x_0 = \sqrt{c - \frac{2c^3}{9A^2}}$

r_0
$= \sqrt{c - \frac{2c^3}{9A^2} + b^2\left(\frac{2c^2}{9A^2}\right)^2}$
$= \sqrt{c - \frac{c^3}{9A^2}}$

$g(c) = c - \frac{c^3}{9A^2}$ とおく.

$\frac{dg(c)}{dc} = 1 - \frac{c^2}{3A^2}$.



$c = \sqrt{3}A$で$g(c)$つまりr_0は最大.

このとき

$\cos\theta$
$= \frac{x_0}{r_0}$
$= \frac{\sqrt{\sqrt{3}A - \frac{2}{3}\sqrt{3}A}}{\sqrt{\sqrt{3}A - \frac{1}{3}\sqrt{3}A}}$
$= \sqrt{\frac{\frac{1}{3}\sqrt{3}A}{\frac{2}{3}\sqrt{3}A}} = \frac{1}{\sqrt{2}}$

(4)

$L(c)$
$= \int_0^{\sqrt{c}} \sqrt{1 + \{f'(x)\}^2}\,dx$

ここで

$\sqrt{1} \leqq \sqrt{1 + \{f'(x)\}^2} \leqq \sqrt{1} + \sqrt{\{f'(x)\}^2}$

$\therefore 1 \leqq \sqrt{1 + \{f'(x)\}^2} \leqq 1 - f'(x)$
$(\because f'(x) \leqq 0)$

$\int_0^{\sqrt{c}} dx \leqq L(c) \leqq \int_0^{\sqrt{c}} (1 - f'(x))dx$
$\Leftrightarrow \sqrt{c} \leqq L(c) \leqq [x - f(x)]_0^{\sqrt{c}}$
$= \sqrt{c} + b$
$\Leftrightarrow 1 \leqq \frac{L(c)}{\sqrt{c}} \leqq 1 + \frac{3A}{2c}$

$\therefore \lim_{c\to\infty} \frac{L(c)}{\sqrt{c}} = 1$

また

$\underbrace{\sqrt{\{f'(x)\}^2}}_{-f'(x)} \leqq \sqrt{1 + \{f'(x)\}^2} \leqq 1 - f'(x)$

より

$\int_0^{\sqrt{c}} (-f'(x))dx \leqq L(c) \leqq \sqrt{c} + b$
$\Leftrightarrow [-f(x)]_0^{\sqrt{c}} \leqq L(c) \leqq \sqrt{c} + \frac{3A}{2\sqrt{c}}$
$\Leftrightarrow \frac{3A}{2\sqrt{c}} \leqq L(c) \leqq \sqrt{c} + \frac{3A}{2\sqrt{c}}$
$\Leftrightarrow \frac{3A}{2} \leqq \sqrt{c}L(c) \leqq c + \frac{3A}{2}$

$\lim_{c\to+0} \sqrt{c}L(c) = \frac{3A}{2}$