2020 中央大(理工)

I

$\overrightarrow{NQ}=\begin{pmatrix}u\\v\\-1\end{pmatrix}$ $\overrightarrow{NP}=\begin{pmatrix}x\\y\\z-1\end{pmatrix}$

$\overrightarrow{NQ}=t\overrightarrow{NP}$ より $-1=t(z-1)$

∴ $1=(1-z)t$

∴ $1-z:1=1:t$ …ⓘ

$t=\frac{1}{1-z}$ より $u=\frac{x}{1-z}$, $v=\frac{y}{1-z}$

$Q\left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right)$

イ…ⓐ

ウ…ⓑ

(CはSとHの交点より)

$\alpha^2+y^2+z^2=1$, $x=\alpha$

また

$u=\frac{\alpha}{1-z} \iff z=\frac{u-\alpha}{u}$

$v=\frac{y}{1-z} \iff (1-z)v=y$

↓

$\frac{\alpha}{u}v=y$

より

$\alpha^2+\frac{\alpha^2}{u^2}v^2+\frac{(u-\alpha)^2}{u^2}=1$

$\iff \alpha^2u^2+\alpha^2v^2+u^2-2\alpha u+\alpha^2=u^2$

$\iff u^2+v^2-2\frac{u}{\alpha}+1=0$

$\iff \left(u-\frac{1}{\alpha}\right)^2+v^2=\frac{1}{\alpha^2}-1=\frac{1-\alpha^2}{\alpha^2}$

エ…ⓗ

オ…ⓘ

$u=\frac{1}{\alpha}+\frac{\sqrt{1-\alpha^2}}{\alpha}\cos\theta$ とおける.

$\min u=\frac{1}{\alpha}-\frac{\sqrt{1-\alpha^2}}{\alpha}$

カ…ⓟ

さらに

$Q_1\left(\frac{\alpha}{1-\sqrt{1-\alpha^2}}, 0, 0\right)$

より

$\overrightarrow{NQ}\cdot\overrightarrow{NQ_1}=\begin{pmatrix}u\\v\\-1\end{pmatrix}\cdot\begin{pmatrix}\frac{\alpha}{1-\sqrt{1-\alpha^2}}\\0\\-1\end{pmatrix}$

$=\frac{u\alpha}{1-\sqrt{1-\alpha^2}}\cdot\frac{1+\sqrt{1-\alpha^2}}{1+\sqrt{1-\alpha^2}}+1$

$=\frac{u(1+\sqrt{1-\alpha^2})}{\alpha}+1$

キ…ⓠ　ク…ⓓ

uの最小値がカのとき つまり

$\cos\theta=-1$ のとき

$P(\alpha, 0, -\sqrt{1-\alpha^2})$ のとき

ケ…ⓐ

コ…ⓒ

サ…ⓛ

最小値

$\frac{1-\sqrt{1-\alpha^2}}{\alpha}\cdot\frac{1+\sqrt{1-\alpha^2}}{\alpha}+1=2$

シ…ⓔ

II

$P_n(x)={}_nC_s\left(\frac{1}{3}\right)^s\left(\frac{2}{3}\right)^{n-s}$

$=\frac{n!\,2^{n-s}}{s!(n-s)!\,3^n}$

ス…ⓒ

$P_n(x-3)={}_nC_{s-1}\left(\frac{1}{3}\right)^{s-1}\left(\frac{2}{3}\right)^{n-s+1}$

$=\frac{n!\,2^{n-s+1}}{(s-1)!(n-s+1)!\,3^n}$

$\frac{P_n(x-3)}{P_n(x)}$

$=\frac{2s!(n-s)!}{(s-1)!(n-s+1)!}=\frac{2s}{n-s+1}\leqq 1$

$P_n(x+3)={}_nC_{s+1}\left(\frac{1}{3}\right)^{s+1}\left(\frac{2}{3}\right)^{n-s-1}$

$=\frac{n!\,2^{n-s-1}}{(s+1)!(n-s-1)!\,3^n}$

$\frac{P_n(x+3)}{P_n(x)}$

$=\frac{n-s}{2(s+1)}\leqq 1$

セ…ⓓ

ソ…ⓢ

$\begin{cases}2s\leqq n-s+1\\ n-s\leqq 2s+2\end{cases} \iff \begin{cases}3s\leqq n+1\\ n-2\leqq 3s\end{cases}$

タ…ⓑ

チ…ⓔ

$Q_n(s)=(3s-n)\frac{n!\,2^{n-s}}{s!(n-s)!\,3^n}$

$\frac{Q_n(s+1)}{Q_n(s)}$

$=\frac{(3s+3-n)n!\,2^{n-s-1}}{(s+1)!(n-s-1)!\,3^n}\cdot\frac{s!(n-s)!\,3^n}{(3s-n)n!\,2^{n-s}}$

$=\frac{(3s+3-n)(n-s)}{(s+1)(3s-n)2}$

ツ…ⓑ

$$\frac{(3s-n+3)(n-s)}{2(3s-n)(s+1)} \leqq 1$$

$\Leftrightarrow (3s-n)(n-s)+3n-3s$

$\leqq (3s-n)(2s+2)$

$\Leftrightarrow 0 \leqq (3s-n)(3s-n+2)$

$-3n+3s$

$\Leftrightarrow (3s-n)^2+6s-2n-3n+3s$

$\geqq 0$

$\Leftrightarrow 9s^2-6ns+n^2+9s-5n \geqq 0$

$\Leftrightarrow \underline{9s^2-3(2n-3)s+n(n-5) \geqq 0}$

テ…ⓓ

$$3s \geqq \frac{2n-3+\sqrt{(2n-3)^2-4n(n-5)}}{2}$$

$$= \underline{n-\frac{3}{2}+\frac{\sqrt{8n+9}}{2}}$$

ト…ⓐ

Lが3の倍数のとき

sが$\frac{L}{3}$または$\frac{L}{3}+1$

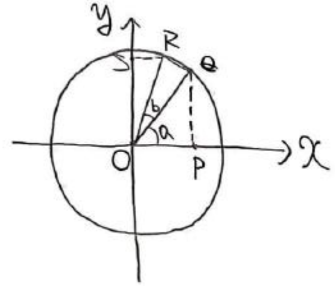
ナ…ⓓ

そうでないとき

sが$\frac{L}{3}$より大きい最小の整数

で$Q_n(s)$は最大. ニ…ⓒ

III



(1)

$A=\triangle OPQ+\triangle ORS$

$=\frac{1}{2}\cos a\sin a$

$\underline{+\frac{1}{2}\sin(a+b)\cos(a+b)}$

(2)

A

$=\frac{1}{4}\sin 2a+\frac{1}{4}\sin 2(a+b)$

$=\frac{1}{4}\{\sin(2a+b-b)+\sin(2a+b+b)\}$

$=\frac{1}{4}\cdot 2\sin(2a+b)\cos b$

$=\frac{1}{2}\sin(2a+b)\cos b$

$0<a<\frac{\pi}{2}-b$より

$b<2a+b<\pi-b$

$\therefore \sin b<\sin(2a+b)\leqq 1$

$\therefore \underline{\frac{1}{2}\sin b\cos b<A\leqq\frac{1}{2}\cos b}$

(3)

$B=A+\frac{1}{2}\sin b$

(2)より

$B\leqq\frac{1}{2}\cos b+\frac{1}{2}\sin b$

$=\frac{\sqrt{2}}{2}\sin\left(b+\frac{\pi}{4}\right)$

つまりBは$b=\frac{\pi}{4}$でかつ

$2a+b=\frac{\pi}{2}$ つまり $a=\frac{\pi}{8}$ で

最大値をとる.

IV

$$f(x)=\frac{1}{x^2}a^{-\frac{1}{x}}$$

(1)

$f'(x)=-2x^{-3}a^{-\frac{1}{x}}$

$+x^{-2}a^{-\frac{1}{x}}(\log a)x^{-2}$

$$=\underline{\frac{a^{-\frac{1}{x}}(-2x+\log a)}{x^4}}$$

(2)

$f'(x)=0 \Leftrightarrow x=\frac{1}{2}\log a$

x	0	$\cdots$	$\frac{1}{2}\log a$	$\cdots$
$f'(x)$		$+$	0	$-$
$f(x)$		$\nearrow$		$\searrow$

$\underline{b=\frac{1}{2}\log a}$

$\boxed{\frac{1}{x}=s\text{とおく}}$

$\lim_{x\to\infty}f(x)=0$

$\lim_{x\to+0}f(x)=\lim_{s\to\infty}s^2\cdot a^{-s}$

$=\lim_{s\to\infty}\frac{1}{s^{-2}a^{s}}$

$=0$

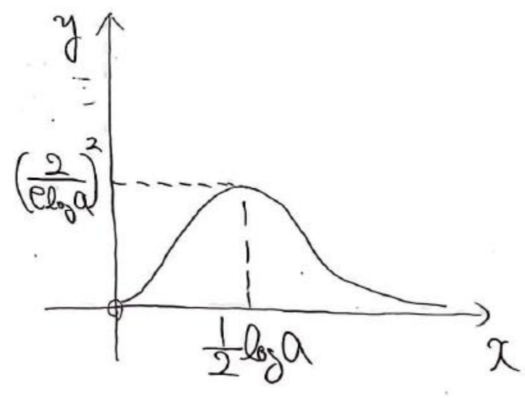
$$f\left(\frac{1}{2}\log a\right)=\frac{a^{-\frac{2}{\log a}}}{\frac{1}{4}(\log a)^2}$$

ここで $\log a^{-\frac{2}{\log a}}=-2$

$\therefore a^{-\frac{2}{\log a}}=e^{-2}$

$$\therefore f\left(\frac{1}{2}\log a\right)=\frac{4e^{-2}}{(\log a)^2}$$

$$=\left(\frac{2}{e\log a}\right)^2$$



(3)

$$l: y=\frac{a^{-\frac{1}{t}}(-2t+\log a)}{t^4}(x-t)+\frac{a^{-\frac{1}{t}}}{t^2}$$

↓原点通る

$$0=\frac{a^{-\frac{1}{t}}(2t^2-t\log a)}{t^4}+\frac{a^{-\frac{1}{t}}}{t^2}$$

$$\Leftrightarrow 0=3-\frac{1}{t}\log a$$

$$\therefore\ \underline{t=\frac{1}{3}\log a}\ _{/\!/}$$

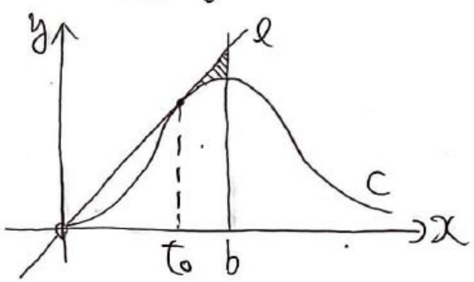
このときlは

$$y=\frac{e^{-3}\left(-\frac{2}{3}\log a+\log a\right)}{\frac{1}{81}(\log a)^4}x$$

$$=\frac{27}{e^3(\log a)^3}x$$

$$=\underline{\left(\frac{3}{e\log a}\right)^3 x}\ _{/\!/}$$

(4) $t_0=\frac{1}{3}\log a$とする



S

$$=\int_{t_0}^{b}\left\{\left(\frac{3}{e\log a}\right)^3x-\frac{1}{x^2}a^{-\frac{1}{x}}\right\}dx$$

$$=\left[\frac{27}{2e^3(\log a)^3}x^2\right]_{t_0}^{b}+\int_{\frac{1}{t_0}}^{\frac{1}{b}}a^{-u}du \qquad \left(u=\frac{1}{x},\ du=-\frac{1}{x^2}dx\right)$$

$$=\frac{27}{2e^3(\log a)^3}(b^2-t_0^2)+\left[-\frac{1}{\log a}a^{-u}\right]_{\frac{1}{t_0}}^{\frac{1}{b}}$$

$$=\frac{27}{2e^3(\log a)}\left(\frac{1}{4}-\frac{1}{9}\right)-\frac{1}{\log a}\left(a^{-\frac{1}{b}}-a^{-\frac{1}{t_0}}\right)$$

$$=\frac{15}{8e^3\log a}-\frac{1}{\log a}\left(a^{-\frac{2}{\log a}}\ a^{-\frac{3}{\log a}}\right)$$

$$=\ \ ''\ \ -\frac{1}{\log a}(e^{-2}-e^{-3})$$

$$=\underline{\frac{23-8e}{8e^3\log a}}\ _{/\!/}$$