2019 日本医科大

[I]

問1 $a=1$のとき連立すると

$$y=(y^2+1)-\frac{3}{4}$$

$$\Leftrightarrow 0=y^2-y+\frac{1}{4}=\left(y-\frac{1}{2}\right)^2$$

$$\therefore y=\frac{1}{2}$$

$$\therefore x^2=\frac{5}{4} \quad \therefore x=\pm\frac{\sqrt{5}}{2}$$

共有点 $\underline{\left(\pm\frac{\sqrt{5}}{2}, \frac{1}{2}\right)}$ //

問2.

連立すると

$$y=a(y^2+1)-\frac{3}{2a+2}$$

$$\Leftrightarrow 0=ay^2-y+a-\frac{3}{2a+2} \cdots ①$$

$$D=(-1)^2-4a\left(a-\frac{3}{2a+2}\right)$$

$$=1-4a^2+\frac{6a}{a+1}>0$$

$$\downarrow \times(a+1)^2$$

$$(a+1)^2-4a^2(a+1)^2+6a(a+1)>0$$

$$\Leftrightarrow (a+1)\{-4a^3-4a^2+7a+1\}>0$$

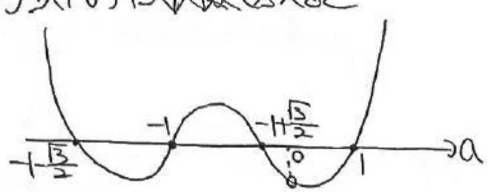
$$\Leftrightarrow (a+1)(4a^3+4a^2-7a-1)<0$$

$$\Leftrightarrow (a+1)(a-1)(4a^2+8a+1)<0$$

$$4a^2+8a+1=0 \Leftrightarrow a=\frac{-4\pm\sqrt{12}}{4}$$

$$=-1\pm\frac{\sqrt{3}}{2}$$

より以下の4次関数で考えると



$$-1-\frac{\sqrt{3}}{2}<a<-1,\ -1+\frac{\sqrt{3}}{2}<a<0,$$

$$0<a<1$$ //

問3、①を解くと

$$y=\frac{1\pm\sqrt{D}}{2a}$$

中心の座標は$(0, \beta)$とおけるので

交点$\left(\pm x_1, \frac{1+\sqrt{D}}{2a}\right), \left(\pm x_2, \frac{1-\sqrt{D}}{2a}\right)$

とおくと、交点と中心の間の距離が

等しいので

$$x_1^2+\left(\frac{1+\sqrt{D}}{2a}-\beta\right)^2=x_2^2+\left(\frac{1-\sqrt{D}}{2a}-\beta\right)^2$$

$$\Leftrightarrow \left(\frac{1+\sqrt{D}}{2a}\right)^2+1+\left(\frac{1+\sqrt{D}}{2a}\right)^2-\frac{1+\sqrt{D}}{a}\beta+\beta^2$$

$$=\left(\frac{1-\sqrt{D}}{2a}\right)^2+1+\left(\frac{1-\sqrt{D}}{2a}\right)^2-\frac{1-\sqrt{D}}{a}\beta+\beta^2$$

$$\Leftrightarrow \frac{2\sqrt{D}}{4a^2}\times 2-\frac{\sqrt{D}}{a}\beta$$

$$=\frac{-2\sqrt{D}}{4a^2}\times 2+\frac{\sqrt{D}}{a}\beta$$

$$\Leftrightarrow \frac{2\sqrt{D}}{a^2}=\frac{2\sqrt{D}}{a}\beta \quad \therefore \beta=\frac{1}{a}$$

中心 $\underline{\left(0, \frac{1}{a}\right)}$ //

$$r^2=2\left(\frac{1+\sqrt{D}}{2a}\right)^2+1-\frac{1+\sqrt{D}}{a^2}+\frac{1}{a^2}$$

$$=\frac{1+D+2\sqrt{D}}{2a^2}+1-\frac{\sqrt{D}}{a^2}$$

$$=\frac{2a^2+1+D}{2a^2}$$

$$=\frac{(2a^2+1)(a+1)+(-4a^3-4a^2+7a+1)}{2a^2(a+1)}$$

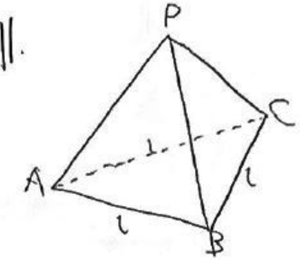
$$=\frac{-2a^3-2a^2+8a+2}{2a^2(a+1)}$$

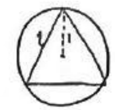
$$=\frac{-a^3-a^2+4a+1}{a^2(a+1)}$$

$$\underline{r=\sqrt{\frac{-a^3-a^2+4a+1}{a^2(a+1)}}}$$ //

[II]

問1.





$\triangle ABC$は半径1の円に内接する

まで、つまり

$$\underline{0<l\leq\sqrt{3}}$$ //

問2.

$\overrightarrow{AH}=\alpha\overrightarrow{AB}+\beta\overrightarrow{AC}$ とおく。

$$\downarrow$$

$$\overrightarrow{AP}\cdot\overrightarrow{AB}=(\overrightarrow{AH}+\overrightarrow{HP})\cdot\overrightarrow{AB}$$

$$=\overrightarrow{AH}\cdot\overrightarrow{AB}$$

$$=\alpha|\overrightarrow{AB}|^2+\beta\overrightarrow{AC}\cdot\overrightarrow{AB}$$

$$=l^2\alpha+\frac{l^2}{2}\beta=s$$

$$\overrightarrow{AP}\cdot\overrightarrow{AC}=(\overrightarrow{AH}+\overrightarrow{HP})\cdot\overrightarrow{AC}$$

$$=\overrightarrow{AH}\cdot\overrightarrow{AC}$$

$$=\frac{l^2}{2}\alpha+l^2\beta=t$$

以上より

$$\frac{3}{2}l^2\alpha=2s-t$$

$$\therefore \alpha=\frac{4s-2t}{3l^2}$$

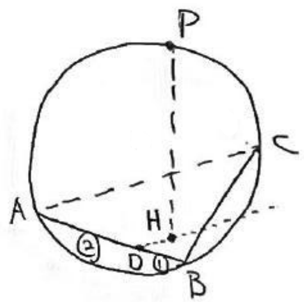
$-\frac{3}{2}t^2\beta = s-2t$

$\therefore \beta = \frac{-2s+4t}{3t^2}$

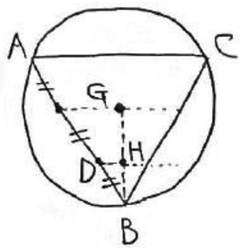
$\therefore \overrightarrow{AH} = \frac{4s-2t}{3t^2}\overrightarrow{AB} + \frac{-2s+4t}{3t^2}\overrightarrow{AC}$

問3. $4s-2t=2t^2$

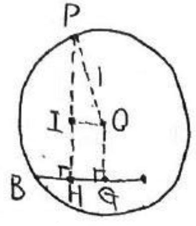
$\overrightarrow{AH} = \frac{2}{3}\overrightarrow{AB} + \frac{-t^2+3t}{3t^2}\overrightarrow{AC}$



ABを2:1に内分する点をDとする.



HはDを通りACに平行な直線上にある. そして△ABCの重心Gに一番近い辺りにある点である.



$BG=\frac{t}{\sqrt{3}}$

三平方

$OG=\sqrt{1^2-\frac{t^2}{3}}$

$PI=\sqrt{1^2-\left(\frac{t}{2\sqrt{3}}\right)^2}=\sqrt{1-\frac{t^2}{12}}$

$\therefore PH=\sqrt{1-\frac{t^2}{12}}+\sqrt{1-\frac{t^2}{3}}$

$\therefore V(t)=\frac{1}{2}t^2\sin60^\circ\times PH\times\frac{1}{3}$

$=\frac{\sqrt{3}t^2}{12}\left(\sqrt{1-\frac{t^2}{12}}+\sqrt{1-\frac{t^2}{3}}\right)$

[IV]

問1.

$f(x)=\log(1+x)-\left(x-\frac{x^2}{2}\right)$ とおく

$f'(x)=\frac{1}{1+x}-1+x$

$=\frac{x^2-1+1}{1+x}=\frac{x^2}{1+x}>0$

よって $f(x)$ は単調増加.

$f(0)=0$ より $f(x)>0$

$g(x)=x-\frac{1}{k+1}\frac{x^2}{2}-\log(1+x)$

$g'(x)=1-\frac{1}{k+1}x-\frac{1}{1+x}$

$=\frac{(k+1)(1+x)-x(1+x)-k-1}{(k+1)(1+x)}$

$=\frac{x(k-x)}{(k+1)(1+x)}\geq0$

$g(x)$ は単調増加.

$g(0)=0$ より $g(x)\geq0$

よって $x-\frac{x^2}{2}\leq\log(1+x)\leq x-\frac{1}{k+1}\frac{x^2}{2}$

問2.

平均値の定理より

$$\begin{cases}\frac{f(a+h)-f(a)}{h}=f'(c) \quad \Leftrightarrow f(a+h)=f(a)+f'(c)h \\ a<c<a+h\end{cases}$$

をみたすcが存在.

$d=\frac{f(a+h)-f(a)}{h}=\frac{2c}{1+c^2}$

$\Leftrightarrow dc^2-2c+d=0$

$\Leftrightarrow c=\frac{1\pm\sqrt{1-d^2}}{d}$

$0<c<1$ より

$c=\frac{1-\sqrt{1-d^2}}{d}$

問3.

$\theta(h)$

$=\frac{\frac{1-\sqrt{1-d^2}}{d}-a}{h}$

$=\frac{1-\sqrt{1-d^2}-ad}{dh}$

$=\frac{(1-ad-\sqrt{1-d^2})(1-ad+\sqrt{1-d^2})}{dh(1-ad+\sqrt{1-d^2})}$

$=\frac{(1-ad)^2-(1-d^2)}{dh(1-ad+\sqrt{1-d^2})}$

$=\frac{-2a+a^2d+d}{h(1-ad+\sqrt{1-d^2})}$

$=\frac{(1+a^2)(d-f'(a))}{h(1-ad+\sqrt{1-d^2})}$

ここで

$\frac{d-f'(a)}{h}$

$=\frac{\frac{f(a+h)-f(a)}{h}-f'(a)}{h}$

$=\frac{\log\{1+(a+h)^2\}-\log(1+a^2)-\frac{2a}{1+a^2}h}{h^2}$

$=\frac{1}{h^2}\left\{\log\left(1+\frac{2ah+h^2}{1+a^2}\right)-\frac{2ah}{1+a^2}\right\}$

ここで問1より $x=k=\dfrac{2ah+h^2}{1+a^2}$ とすると

$$\frac{2ah+h^2}{1+a^2}-\frac{1}{2}\left(\frac{2ah+h^2}{1+a^2}\right)^2 \leqq \log\left(1+\frac{2ah+h^2}{1+a^2}\right) \leqq \frac{2ah+h^2}{1+a^2}-\frac{1}{2(k+1)}\left(\frac{2ah+h^2}{1+a^2}\right)^2$$

$$\Leftrightarrow \frac{h^2}{1+a^2}-\frac{1}{2}\left(\frac{2ah+h^2}{1+a^2}\right)^2 \leqq \log\left(1+\frac{2ah+h^2}{1+a^2}\right)-\frac{2ah}{1+a^2} \leqq \frac{h^2}{1+a^2}-\frac{1}{2(k+1)}\left(\frac{2ah+h^2}{1+a^2}\right)^2$$

$$\Leftrightarrow \frac{1}{1+a^2}-\frac{1}{2}\left(\frac{2a+h}{1+a^2}\right)^2 \leqq \frac{1}{h^2}\left\{\log\left(1+\frac{2ah+h^2}{1+a^2}\right)-\frac{2ah}{1+a^2}\right\} \leqq \frac{1}{1+a^2}-\frac{1}{2(k+1)}\left(\frac{2a+h}{1+a^2}\right)^2$$

↓ $h\to 0+0$ のとき $k\to 0+0$ から

$$\lim_{h\to 0+0}\frac{d-f(a)}{h}=\frac{1}{1+a^2}-\frac{(2a)^2}{2(1+a^2)^2}=\frac{1-a^2}{(1+a^2)^2}$$

つまり

$$\lim_{\theta\to 0+0}\theta(h)=(1+a^2)\cdot\frac{1-a^2}{(1+a^2)^2}\cdot\frac{1}{1-af'(a)+\sqrt{1-\{f'(a)\}^2}}$$

$$=\frac{1-a^2}{1+a^2}\cdot\frac{1}{1-\dfrac{2a^2}{1+a^2}+\dfrac{1-a^2}{1+a^2}}$$

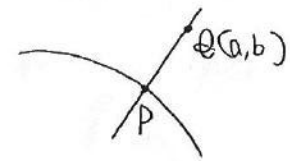
$$=\frac{1-a^2}{1+a^2-2a^2+1-a^2}$$

$$=\frac{1}{2}$$

[IV]

問1. CのP$(t, f(t))$における法線は

$$y=-\frac{1}{f'(t)}(x-t)+f(t)$$



直線PQの方向ベクトルは $\begin{pmatrix}1\\ -\dfrac{1}{f'(t)}\end{pmatrix}$

$|\overrightarrow{PQ}|=1$ より

$$\overrightarrow{PQ}=\frac{1}{\sqrt{1+\dfrac{1}{\{f'(t)\}^2}}}\begin{pmatrix}1\\ -\dfrac{1}{f'(t)}\end{pmatrix}=\begin{pmatrix}\dfrac{f'(t)}{\sqrt{\{f'(t)\}^2+1}}\\ -\dfrac{1}{\sqrt{\{f'(t)\}^2+1}}\end{pmatrix}$$

問2

$\overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{PQ}$ より

$$a=t+\frac{f'(t)}{\sqrt{\{f'(t)\}^2+1}}$$

$$b=f(t)-\frac{1}{\sqrt{\{f'(t)\}^2+1}}$$

問3.

$$b'(t)=f'(t)-\{[f'(t)]^2+1\}^{-\frac{3}{2}}\left(-\frac{1}{2}\right)2f'(t)f''(t)$$

$$=f'(t)+\frac{f'(t)f''(t)}{\{[f'(t)]^2+1\}^{\frac{3}{2}}}$$

$$a'(t)=1+\frac{f''(t)\sqrt{\{f'(t)\}^2+1}-f'(t)\frac{1}{2}\{[f'(t)]^2+1\}^{-\frac{1}{2}}\,2f'(t)f''(t)}{[f'(t)]^2+1}$$

$$=1+\frac{f''(t)}{\{[f'(t)]^2+1\}^{\frac{3}{2}}}$$

より $b'(t)=f'(t)a'(t)$ が成立.

問4.

$$\Delta L(T_1,T_2)$$

$$=L_Q(T_1,T_2)-L_P(T_1,T_2)$$

$$=\int_{T_1}^{T_2}\sqrt{\{a'(t)\}^2+\{b'(t)\}^2}\,dt-\int_{T_1}^{T_2}\sqrt{1+\{f'(t)\}^2}\,dt$$

$$=\int_{T_1}^{T_2}(a'(t)-1)\sqrt{1+\{f'(t)\}^2}\,dt$$

$$=\int_{T_1}^{T_2}\frac{f''(t)}{\{[f'(t)]^2+1\}^{\frac{3}{2}}}\sqrt{1+\{f'(t)\}^2}\,dt$$

$$=\int_{T_1}^{T_2}\frac{f''(t)}{1+\{f'(t)\}^2}\,dt$$

問5

$$f(x)=\frac{e^{x}+e^{-x}}{2}=\cosh x$$
$$f'(x)=\frac{e^{x}-e^{-x}}{2}=\sinh x$$

とおくと

$$\cosh^2 x-\sinh^2 x=1.$$

$$\frac{f'(t)}{1+\{f'(t)\}^2}=\frac{\cosh t}{1+\sinh^2 t}$$
$$=\frac{\cosh t}{\cosh^2 t}$$
$$=\frac{1}{\cosh t}$$

$\Delta L(T_1,T_2)$

$$=\int_{T_1}^{T_2}\frac{2}{e^{t}+e^{-t}}dt$$
$$=\int_{T_1}^{T_2}\frac{2e^{t}}{e^{2t}+1}dt \qquad e^{t}=u$$

$\downarrow$ $e^{t}=u$ とおく $e^{t}dt=du$

$$=\int_{e^{T_1}}^{e^{T_2}}\frac{2}{u^2+1}du$$

$\downarrow$ $u=\tan\theta$, $du=\frac{1}{\cos^2\theta}d\theta$

$$\begin{array}{c|c} u & e^{T_1}\to e^{T_2} \\ \hline \theta & \theta_1\to\theta_2 \end{array}$$

$$=\int_{\theta_1}^{\theta_2}\frac{2}{\tan^2\theta+1}\cdot\frac{1}{\cos^2\theta}d\theta$$
$$=2(\theta_2-\theta_1)$$

ここで $T_1\to-\infty$, $T_2\to+\infty$ とすると

$\theta_1\to+0$, $\theta_2\to\frac{\pi}{2}$ より

$$\lim_{T_1\to-\infty,\,T_2\to+\infty}\Delta L(T_1,T_2)=2\left(\frac{\pi}{2}-0\right)=\pi$$ //