2018 慶應 (医)

[I]

(1)

$\log_a(x+2a) - \log_{\frac{1}{a}}\left(\frac{x}{4}\right)$

$= \log_a(x+2a) + \log_a \frac{x}{4}$

$= \log_a \frac{x}{4}(x+2a) < \log_a a^2$

$\Leftrightarrow \frac{x}{4}(x+2a) < a^2$

$\Leftrightarrow x^2+2ax-4a^2<0$

$\Leftrightarrow -a-\sqrt{5a^2} < x < a+\sqrt{5a^2}$

$\therefore (1-\sqrt{5})a < x < (1+\sqrt{5})a$

ところで真数条件は

$\begin{cases} x+2a>0 \\ \frac{x}{4}>0 \end{cases}$ $\therefore x>0$

まとめると

$0 < x < (1+\sqrt{5})a$

$A(a) \subset (-\infty, 4)$ であるためには

$(\sqrt{5}-1)a \leqq 4$

$\Leftrightarrow a \leqq \frac{4}{\sqrt{5}-1} = \sqrt{5}+1$

$\therefore \underline{1 < a \leqq \sqrt{5}+1}$ //

(2)

$2\vec{a}+3\vec{b} = -4\vec{c}$

↓大きさの2乗

$4|\vec{a}|^2 + 12\vec{a}\cdot\vec{b} + 9|\vec{b}|^2 = 16|\vec{c}|^2$

$\therefore \underline{\vec{a}\cdot\vec{b} = \frac{1}{4}}$ //

$3\vec{b}+4\vec{c} = -2\vec{a}$

↓大きさの2乗

$9|\vec{b}|^2 + 24\vec{b}\cdot\vec{c} + 16|\vec{c}|^2 = 4|\vec{a}|^2$

$\therefore \vec{b}\cdot\vec{c} = -\frac{21}{24} = -\frac{7}{8}$

$4\vec{c}+2\vec{a} = -3\vec{b}$

↓大きさの2乗

$16|\vec{c}|^2 + 16\vec{c}\cdot\vec{a} + 4|\vec{a}|^2 = 9|\vec{b}|^2$

$\therefore \vec{c}\cdot\vec{a} = -\frac{11}{16}$

$|\vec{a}+\vec{b}+t\vec{c}|^2$

$= 1+1+t^2+2\cdot\frac{1}{4}+\left(-\frac{7t}{4}\right)+2t\left(-\frac{11}{16}\right)$

$= t^2 - \frac{25}{8}t + \frac{5}{2}$

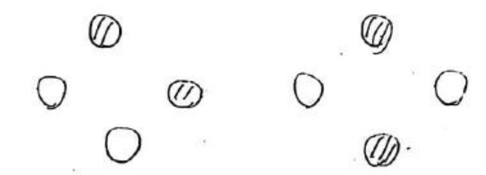
$= \left(t-\frac{25}{16}\right)^2 + \frac{15}{256}$

$t=\frac{25}{16}$ のとき

$\underline{最小値 \frac{\sqrt{15}}{16}}$ //

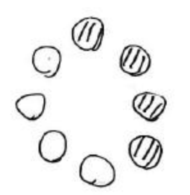
(3)

$k=1$ のとき

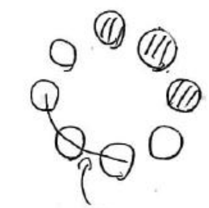


$\underline{N_1 = 2}$ //

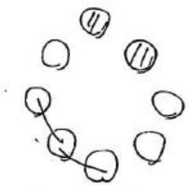
$k=2$ のとき



1とおり

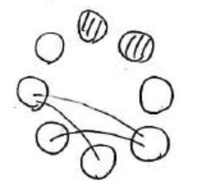


3とおり



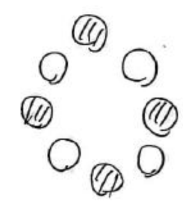
赤が2コずつ隣りあう

2とおり



赤2コだけ隣りあう残りは隣りあわない)

3とおり



1とおり

以上より

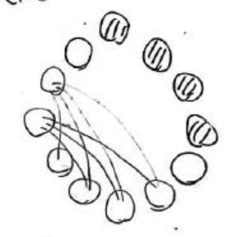
$\underline{N_2 = 10}$ //

$k=3$ のとき

(i) 赤が6コ連続 … 1とおり

(ii) 赤が5コ連続 … 5とおり

(iii)

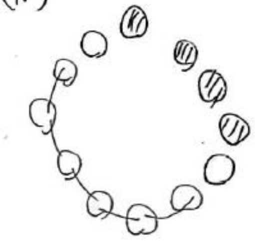


赤4コ連続のみ

$4+3+2+1$

$=10$

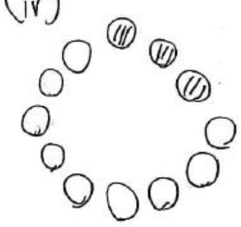
(iv)



赤4コ, 赤2コ連続

5

(v)



赤3コ連続のみ

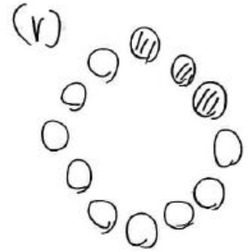
$3+2+1$

$+2+1$

$+1$

$=10$

(v)

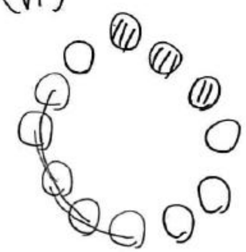


赤3コ, 赤2コ連続

$(4+3+3)\times 2$

$=20$

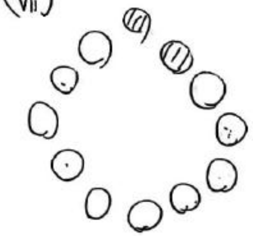
(vi)



赤3コ, 赤3コ連続

3とおり

(vii)



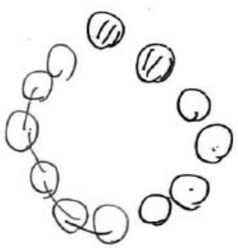
赤2コ連続のみ

2+1

+1

+1

(viii)



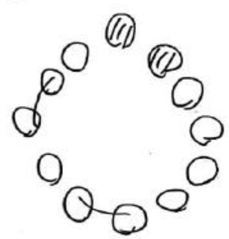
赤2コ, 赤2コ

連続のみ

6+3+(1+3)+2

= 15

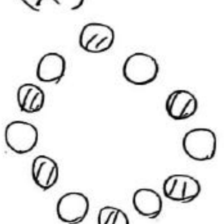
(ix)



赤2コ, 3組連続

4とおり

(x)



1コのみ　1とおり

すべてたすと

$N_3 = \underline{80}$ //

[II]

(1)

n回目　　n+1回目

$$B に3枚 \quad a_n(k) \xrightarrow{1-\frac{2}{3k}} a_{n+1}(k)$$

$$a_n(k) \xrightarrow{\frac{2}{3k}} b_{n+1}(k)$$

$$B に2枚 \quad b_n(k) \xrightarrow{1-\frac{1}{2}\cdot\frac{1}{k+1}} b_{n+1}(k)$$

$$\underline{a_n(k) = \left(1-\frac{2}{3k}\right)a_{n-1}(k)} \; //$$

$$b_n(k) = \underline{\frac{2}{3k}}a_{n-1}(k) + \underline{\left(1-\frac{1}{2k+2}\right)}b_{n-1}(k) \; //$$

(2)

$$a_n(k) = \left(1-\frac{2}{3k}\right)^n \underbrace{a_0(k)}_{1}$$

$$= \left(1-\frac{2}{3k}\right)^n$$

$$\lim_{k\to\infty} a_k(k)$$

$$= \lim_{k\to\infty}\left(1-\frac{2}{3k}\right)^k$$

$$-\frac{2}{3k} = t$$

$$\Leftrightarrow -\frac{2}{3t} = k$$

$$= \lim_{t\to 0}\left\{(1+t)^{\frac{1}{t}}\right\}^{-\frac{2}{3}}$$

$$= \underline{e^{-\frac{2}{3}}} \; //$$

(3) $k=3$のとき

$$a_n(3) = \left(\frac{7}{9}\right)^n$$

$$b_n(3) = \frac{2}{9}a_{n-1}(3) + \frac{7}{8}b_{n-1}(3)$$

$$= \frac{7}{8}b_{n-1}(3) + \frac{2}{9}\left(\frac{7}{9}\right)^{n-1}$$

$$\left(\frac{9}{7}\right)^n b_n(3)$$

$$= \frac{9}{8}\left(\frac{9}{7}\right)^{n-1}b_{n-1}(3) + \frac{2}{7}$$

$$\downarrow \left(\frac{9}{7}\right)^n b_n(3) = c_n$$

$$c_n = \frac{9}{8}c_{n-1} + \frac{2}{7}$$

(特)　$-\frac{1}{8}\alpha = \frac{2}{7}$

$$\Leftrightarrow \alpha = -\frac{16}{7}$$

$$c_n + \frac{16}{7} = \frac{9}{8}\left(c_{n-1} + \frac{16}{7}\right)$$

$$\downarrow$$

$$c_n + \frac{16}{7} = \left(\underbrace{c_1}_{\frac{2}{7}} + \frac{16}{7}\right)\left(\frac{9}{8}\right)^{n-1}$$

$$\Leftrightarrow c_n = \frac{18}{7}\cdot\left(\frac{9}{8}\right)^{n-1} - \frac{16}{7}$$

$$\therefore b_n(3)$$

$$= \frac{18}{7}\left(\frac{9}{8}\right)^{n-1}\frac{7^n}{9^n} - \frac{16}{7}\left(\frac{7}{9}\right)^n$$

$$= \frac{18}{7}\cdot\frac{8}{9}\left(\frac{9}{8}\right)^n\left(\frac{7}{9}\right)^n - \frac{16}{7}\left(\frac{7}{9}\right)^n$$

$$= \frac{16}{7}\left(\frac{7}{8}\right)^n - \frac{16}{7}\left(\frac{7}{9}\right)^n$$

$$= \underline{\frac{16}{7}\left(\frac{7}{8}\right)^n\left\{1-\left(\frac{8}{9}\right)^n\right\}} \; //$$

[III]

(1)

$$\cos(3x+x)=\cos 3x\cos x-\sin 3x\sin x$$
$$-)\ \cos(3x-x)=\cos 3x\cos x+\sin 3x\sin x$$
$$\cos 4x-\cos 2x=-2\sin 3x\sin x$$
$$\Leftrightarrow \sin 3x\sin x=\frac{1}{2}(\cos 2x-\cos 4x)$$

$f(x)$
$$=\frac{1}{2}\sin 2x(\cos 2x-\cos 4x)$$
$$=\frac{1}{4}\sin 4x-\frac{1}{2}\sin 2x\cos 4x$$

$$\sin(4x+2x)=\sin 4x\cos 2x+\cos 4x\sin 2x$$
$$-)\ \sin(4x-2x)=\sin 4x\cos 2x-\cos 4x\sin 2x$$
$$\sin 6x-\sin 2x=+2\sin 2x\cos 4x$$

よって

$f(x)$
$$=\frac{1}{4}\sin 4x-\frac{1}{2}\cdot\frac{1}{2}(\sin 6x-\sin 2x)$$
$$=\frac{1}{4}\sin 2x+\frac{1}{4}\sin 4x-\frac{1}{4}\sin 6x$$

$p=\frac{\pi}{3}$ より

(ii)
$$=\int_0^{\frac{\pi}{3}} f(x)\,dx$$
$$=\left[-\frac{1}{8}\cos 2x-\frac{1}{16}\cos 4x+\frac{1}{24}\cos 6x\right]_0^{\frac{\pi}{3}}$$
$$=\frac{1}{16}+\frac{1}{32}+\frac{1}{24}-\left(-\frac{1}{8}-\frac{1}{16}+\frac{1}{24}\right)$$
$$=\frac{2+1+4+2}{32}=\frac{9}{32}$$

(2)

$$\sin 2x\sin x=\frac{1}{2}(\cos x-\cos 3x)$$
$$\sin 3x\sin 2x=\frac{1}{2}(\cos x-\cos 5x)$$
$$\sin 3x\sin x=\frac{1}{2}(\cos 2x-\cos 4x)$$

$g(x)$
$$=\frac{1}{2}\cos 3x-\frac{1}{2}\cos 5x-\frac{1}{2}\cos 2x+\frac{1}{2}\cos 4x$$
$$=-\frac{1}{2}(\cos 2x+\cos 5x)+\frac{1}{2}\cos 3x+\frac{1}{2}\cos 4x$$

$g\left(\frac{\pi}{7}\right)$
$$=-\frac{1}{2}\left(\cos\frac{2\pi}{7}+\cos\frac{5\pi}{7}\right)+\frac{1}{2}\left(\cos\frac{3\pi}{7}+\cos\frac{4\pi}{7}\right)$$
$$=0$$

(3)

A^2
$$=\sin^2\alpha+\sin^2\beta+\sin^2\gamma+2C$$
$$=\frac{1-\cos 2\alpha}{2}+\frac{1-\cos 2\beta}{2}+\frac{1-\cos 2\gamma}{2}+2C$$
$$=\frac{3}{2}-\frac{1}{2}B+2C$$
$$\Leftrightarrow C=\frac{1}{2}A^2+\frac{1}{4}B-\frac{3}{4}\ \cdots☆$$

$$\sum_{k=1}^{7} z^k$$
$$=\frac{z-z^8}{1-z}=\frac{z-z}{1-z}=0$$
$$\Downarrow$$
$$\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}+\cos\frac{4\pi}{7}+i\sin\frac{4\pi}{7}+\cos\frac{6\pi}{7}+i\sin\frac{6\pi}{7}+\cos\frac{8\pi}{7}+i\sin\frac{8\pi}{7}+\cos\frac{10\pi}{7}+i\sin\frac{10\pi}{7}+\cos\frac{12\pi}{7}+i\sin\frac{12\pi}{7}+1$$
$$=2\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}\right)+1=0$$
$$\therefore B=-\frac{1}{2}$$

☆より
$$\underbrace{g\left(\frac{\pi}{7}\right)}_{0}=\frac{1}{2}A^2-\frac{1}{8}-\frac{3}{4}$$
$$\Leftrightarrow 0=4A^2-7$$
$$\therefore A^2=\frac{7}{4}$$

$\alpha=\frac{\pi}{7},\ \beta=-\frac{2\pi}{7},\ \gamma=-\frac{3\pi}{7}$ より

$A<0$ なので
$$A=-\frac{\sqrt{7}}{2}$$

$f\left(\frac{\pi}{7}\right)$
$$=\frac{1}{4}\sin\frac{2\pi}{7}+\frac{1}{4}\sin\frac{4\pi}{7}-\frac{1}{4}\sin\frac{6\pi}{7}$$
$$=-\frac{1}{4}\sin\frac{\pi}{7}-\frac{1}{4}\sin\left(-\frac{2\pi}{7}\right)-\frac{1}{4}\sin\left(-\frac{3\pi}{7}\right)$$
$$=-\frac{1}{4}A$$
$$=\frac{\sqrt{7}}{8}$$

[IV]

(1)

$x=\theta+\sin\theta \qquad y=1-\cos\theta$

$\dfrac{dx}{d\theta}=1+\cos\theta \qquad \dfrac{dy}{d\theta}=\sin\theta$

$\downarrow$

$\dfrac{dy}{dx}=\dfrac{\sin\theta}{1+\cos\theta}$

$\theta=0$のとき $\dfrac{dy}{dx}=0$ かつ $y=0$.

Cは$(0,0)$でx軸に接する.

$\dfrac{dx}{d\theta}>0$ より xは単調増加.

$\underline{-\pi}<x<\underline{\pi}$

$y=1-\cos\theta$より $\underline{0\leqq}\, y<\underline{2}$

(2)

$\dfrac{dx}{d\theta}=1+\cos\theta>0 \quad (-\pi<\theta<\pi)$

よりxは単調増加するので

任意の$x(-\pi<x<\pi)$に対して

C上の点は1つに定まる.

また

$\dfrac{d^2y}{dx^2}=\dfrac{\cos\theta(1+\cos\theta)+\sin^2\theta}{(1+\cos\theta)^2}\cdot\dfrac{d\theta}{dx}$

$=\dfrac{\cos\theta+1}{(1+\cos\theta)^3}$

$=\dfrac{1}{(\cos\theta+1)^2}>0$

よって下に凸.

(3)

$L(t)$

$=\displaystyle\int_0^t\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$

$=\displaystyle\int_0^t\sqrt{1+2\cos\theta+\cos^2\theta+\sin^2\theta}\,d\theta$

$=\displaystyle\int_0^t\sqrt{2+2\cos\theta}\,d\theta$

$=\displaystyle\int_0^t\sqrt{2+2\left(2\cos^2\frac{\theta}{2}-1\right)}\,d\theta$

$=\displaystyle\int_0^t\left|2\cos\frac{\theta}{2}\right|d\theta$

$=\displaystyle\int_0^t 2\cos\frac{\theta}{2}\,d\theta$

$=\left[4\sin\dfrac{\theta}{2}\right]_0^t$

$=\underline{4\sin\dfrac{t}{2}}$

(4)

$l_t: y=\dfrac{\sin t}{1+\cos t}(x-t-\sin t)+1-\cos t$

$\downarrow y=0$

$0=\sin t(x-t-\sin t)+1-\cos^2 t$

$\Leftrightarrow 0=(\sin t)x-t\sin t$

$\therefore\ \underline{x=t}$ ← Q_tのx座標

$Q_t(t,0)$, $P_t(t+\sin t, 1-\cos t)$

$\overrightarrow{Q_tP_t}=\begin{pmatrix}\sin t\\ 1-\cos t\end{pmatrix}\cdot \overrightarrow{e_1}=\begin{pmatrix}1\\0\end{pmatrix}$

とすると求める角をφとすれば

$\overrightarrow{Q_tP_t}\cdot\overrightarrow{e_1}=|\overrightarrow{Q_tP_t}|\cdot|\overrightarrow{e_1}|\cos\varphi$

$\Leftrightarrow \sin t=\sqrt{2-2\cos t}\cos\varphi \cdots ①$

ここで

$\sqrt{2-2\cos t}=\sqrt{2-2\left(1-2\sin^2\frac{t}{2}\right)}$

$=\sqrt{4\sin^2\frac{t}{2}}$

$=\left|2\sin\frac{t}{2}\right|$

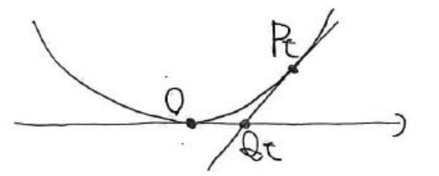
$=2\sin\frac{t}{2}$

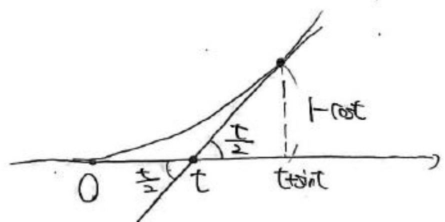
①は

$2\sin\frac{t}{2}\cos\frac{t}{2}=2\sin\frac{t}{2}\cos\varphi$

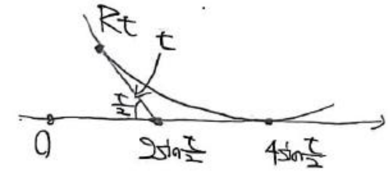
$\therefore\ \underline{\varphi=\dfrac{t}{2}}$

(5)





転がすと



$a(t)=2\sin\frac{t}{2}-t\cos\frac{t}{2}$

$b(t)=t\sin\frac{t}{2}$

(6) $a(t)=X$ $b(t)=Y$とする.

$\dfrac{dX}{dt}=\cos\frac{t}{2}-\cos\frac{t}{2}+\frac{t}{2}\sin\frac{t}{2}$

(い)

$=\displaystyle\int_0^a Y\,dX$

$=\displaystyle\int_0^\pi t\sin\frac{t}{2}\cdot\frac{t}{2}\sin\frac{t}{2}\,dt$

$=\dfrac{1}{2}\displaystyle\int_0^\pi t^2\sin^2\frac{t}{2}\,dt$

$=\dfrac{1}{2}\displaystyle\int_0^\pi t^2\,\frac{1-\cos t}{2}\,dt$

$=\dfrac{1}{4}\left[\dfrac{t^3}{3}-t^2\sin t-2t\cos t+2\sin t\right]_0^\pi$

$-t^2\cos t-2t\sin t+2t\sin t-2\cos t+2\cos t$

$=\underline{\dfrac{\pi^3}{12}+\dfrac{\pi}{2}}$